Title	On a semilinear elliptic equation with subcritical exponent in higher dimensional space (Variational Problems and Related Topics)
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Citation	数理解析研究所講究録 (2009), 1671: 96-112
Issue Date	2009-12
URL	http://hdl.handle.net/2433/141164
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1671 巻 2009 年 96-112

On a semilinear elliptic equation with subcritical exponent in higher dimensional space

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Abstract

We study some properties of the solution to a semilinear elliptic equation with subcritical expenent in higher dimensions. Classification of the bounded energy solution in whole space, an inequality of sup + inf type, a theorem of Brezis-Merle type, and the quantized blowup mechanism are presented.

1 Introduction

In this paper, we study the semilinear elliptic equation

$$\begin{cases} -\Delta v = v_+^\gamma & \text{in } \Omega \\ \int_\Omega v_+^{\frac{n(\gamma-1)}{2}} dx < +\infty, \end{cases} \tag{1.1}$$

where $\gamma \in \left(1, \frac{n+2}{n-2}\right)$, $n \geq 3$, and $\Omega \subset \mathbf{R}^n$ is a bouned domain with smooth boundary $\partial\Omega$ or $\Omega = \mathbf{R}^n$. In the case $\gamma = \frac{n}{n-2}$, classification of the solution to (1.1) with $\Omega = \mathbf{R}^n$, inequalities of $\sup + \inf$ and Trudinger-Moser type, and blowup analysis of the solution are done in [21]. As stated there, equation (1.1) is close to Liouville's equation in two dimensions,

$$\begin{cases} -\Delta v = e^v & \text{in } \Omega \subset \mathbf{R}^2 \\ \int_\Omega e^v dx < +\infty. \end{cases} \tag{1.2}$$

In fact, equations (1.1) and (1.2) have the following common properties:

(A) Scaling invariance concerning the equation and the energy

(B) Classification of the bounded energy solution in whole space

(C) Existence of a $\sup + \inf$ type inequality

(D) Alternatives concerning convergence of the solutions

(E) Quantized blowup mechanism

In what follows, we look over these properties.

(A) For a solution $v = v(x)$ to (1.2), the transformation $v_\mu(x) = v(\mu x) + 2\log\mu$, $\mu > 0$, satisfies

$$\begin{cases} -\Delta v_\mu = e^{v_\mu} & \text{in } \Omega_\mu \\ \int_{\Omega_\mu} e^{v_\mu} dx = \int_\Omega e^v dx, \end{cases}$$

where $\Omega_\mu = \{y \in \mathbf{R}^2 \mid \mu y \in \Omega\}$. Similarly, for a solution $v = v(x)$ to (1.1), the transformation $v_\mu(x) = \mu^q v(\mu x)$, $\mu > 0$, $q = \frac{2}{\gamma - 1}$, satisfies

$$\begin{cases} -\Delta v_\mu = (v_\mu)_+^\gamma & \text{in } \Omega_\mu \\ \int_{\Omega_\mu} (v_\mu)_+^{\frac{n(\gamma-1)}{2}} dx = \int_\Omega v_+^{\frac{n(\gamma-1)}{2}} dx, \end{cases}$$

where $\Omega_\mu = \{y \in \mathbf{R}^n \mid \mu y \in \Omega\}$, $n \geq 3$. These scale invariances are important extremely in the proof of the properties (B)-(E), and, in particular, allow us to the blowup analysis and the hierarchical argument.

(B) Any nontrivial classical solution to (1.2) in whole space (i.e., $\Omega = \mathbf{R}^2$) has the form

$$v(x) = \log\left\{\frac{8\mu^2}{(1 + \mu^2|x - x_0|^2)}\right\} \tag{1.3}$$

for some $x_0 \in \mathbf{R}^2$. This fact is shown by Chen and Li [4]. Similar fact for (1.1) with $\gamma = \frac{n}{n-2}$ is done by Wang and Ye [21]. A crucial difference between (1.3) and (1.4) below is whether a support of the positive part of the solution is compact or not. This makes several arguments for (1.1) simpler. We now state the first result.

Theorem 1 *Assume that $\gamma \in \left(1, \frac{n+2}{n-2}\right)$ and $n \geq 3$. Then, any non-constant classical solution $v = v(x)$ to (1.1) with $\Omega = \mathbf{R}^n$ is radially symmetric, and the nonnegative part v_+ has a compact support. More precisely, there exist $x_0 \in \mathbf{R}^n$ and $\mu > 0$ such that*

$$v(x) = \begin{cases} \mu^q \phi(\mu|x - x_0|) & (\mu|x - x_0| \leq r_\gamma^*) \\ \frac{\lambda_\gamma^*}{\omega_{n-1}(n-2)}\left(\frac{1}{|x - x_0|^{n-2}} - \frac{1}{(\mu^{-1} r_\gamma^*)^{n-2}}\right) & (\mu|x - x_0| > r_\gamma^*) \end{cases} \tag{1.4}$$

with ω_{n-1} standing for the area of the boundary of the unit ball in $\mathbf{R}^n$, where r_γ^* is the first zero point of the unique solution $\phi = \phi(r)$ to

$$\begin{cases} \phi''(r) + \frac{n-1}{r}\phi'(r) + \phi_+^\gamma(r) = 0, & r > 0 \\ \phi(0) = 1, \quad \phi'(0) = 0, \end{cases} \tag{1.5}$$

and

$$\lambda_\gamma^* = \omega_{n-1} \int_0^{r_\gamma^*} \phi^{\frac{n(\gamma-1)}{2}} r^{n-1} dr. \tag{1.6}$$

The general entire solution to

$$-\Delta v = v^p \qquad \text{in } \mathbf{R}^n,\ n \geq 3 \tag{1.7}$$

is concerned with the critical Sobolev exponent, i.e., $p_s = \frac{n-2}{n+2}$. Gidas and Spruck showed [8] that there is no positive solution to (1.7) in subcritical case $1 \le p < p_s$. On the other hand, it was shown by Caffarelli, Gidas, and Spruck [3] that (1.7) has the positive solutions in critical case $p = p_s$. Furthermore, the solution to $v = v(x)$ to (1.7) with $p = p_s$ has the form

$$v(x) = \frac{\{n(n-2)\mu^2\}^{\frac{n-2}{4}}}{(\mu^2 + |x - x_0|^2)^{\frac{n-2}{2}}}$$

for some $x_0 \in \mathbf{R}^n$ and $\mu > 0$ if $v(x) = O(|x|^{2-n})$ as $|x| \to +\infty$. In super critical case $p > p_s$, radial symmetry of the positive solution to (1.7) no longer hold generally, see [11, 22] for details.

(C) The $\sup + \inf$ type inequality for (1.2) was shown by Shafrir [16], see also [2, 6]. Several $\sup \times \inf$ type inequalities for equations concerning the critical Sobolev exponent are found in [5, 12, 14]. The inequality of $\sup + \inf$ type for (1.1) with $\gamma = \frac{n}{n-2}$ was established in [21]. We extend it to the case $\gamma \in \left(1, \frac{n+2}{n-2}\right)$.

Theorem 2 *Assume that $\gamma \in \left(1, \frac{n+2}{n-2}\right)$ and $n \ge 3$. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain. Then, for any compact set $K \subset \Omega$ and any number $T > 0$, there exist $C_1 = C_1(n, \gamma) > 0$ and $C_2 = C_2(n, \gamma, K, T) > 0$ such that*

$$\sup_K v + C_1 \inf_\Omega v \le C_2 \tag{1.8}$$

for any solution $v = v(x)$ to (1.1) with the property

$$\int_\Omega v_+^{\frac{n(\gamma-1)}{2}} dx \le T. \tag{1.9}$$

(D) Convergence of the solutions to (1.2) was studied by Brezis and Merle [1], and then the stronger result was obtained by Li and Shafrir [13]. We note that the $\sup + \inf$ type inequality is a crucial component of the proof of the latter result, see [13]. The corresponding results for (1.1) with $\gamma = \frac{n}{n-2}$ are shown in [21]. They are extend as follows.

Theorem 3 *Assume that $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ and $n \ge 3$. Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $\{v_k\}$ be a sequence of the classical solutions satisfying*

$$\begin{cases} -\Delta v_k = (v_k)_+^\gamma & \text{in } \Omega \\ \int_\Omega (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \le T \end{cases} \tag{1.10}$$

for some $T > 0$. Then there exists a subsequence, still denoted by the same symbol $\{v_k\}$, such that the following alternatives occur:

(i) *$\{v_k\}$ is locally uniformly bounded.*
(ii) *$v_k \to -\infty$ locally uniformly in Ω.*

(iii) *There exists a finite set* $\mathcal{S} = \{x_i\}_{i=1}^m$ *such that* $v_k \to -\infty$ *locally uniformly in* $\Omega \setminus \mathcal{S}$ *and that*

$$(v_k)_+^{\frac{n(\gamma-1)}{2}} dx \rightharpoonup \sum_{i=1}^m \alpha_*(x_i)\delta_{x_i}(dx)$$

in $\mathcal{M}(\Omega)$ *with* $\alpha_*(x_i) = l_i\lambda_\gamma^*$ *for some* $l_i \in \mathbf{N}$ *and for all* $i = 1, \cdots, m$*, where* δ_{x_i} *and* $\mathcal{M}(\Omega)$ *denote the Dirac measure and the space of measure, respectively, and* λ_γ^* *is as in (1.6).*

(E) Nagasaki and Suzuki [15] studied the quantized blowup mechanism for

$$\begin{cases} -\Delta v = \sigma e^v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

The result is applicapable for

$$\begin{cases} -\Delta w = e^w & \text{in } \Omega \\ w = \text{(unknown) constant} & \text{on } \partial\Omega \\ \int_\Omega e^w dx = \lambda \end{cases} \tag{1.11}$$

by combining the results by [1, 13, 7]. Then the quantized blowup mechanism also arises for (1.11), see [19] for details. Here, we consider

$$\begin{cases} -\Delta v = v_+^\gamma & \text{in } \Omega \\ v = \text{(unknown) constant} & \text{on } \partial\Omega \\ \int_\Omega v^{\frac{n(\gamma-1)}{2}} dx = \lambda. \end{cases} \tag{1.12}$$

The corresponding result for $\gamma = \frac{n}{n-2}$ is shown in [19]. This property holds even in the case $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$.

Theorem 4 *Assume that* $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ *and* $n \geq 3$*. Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain with smooth boundary* $\partial\Omega$*, and* (λ_k, v_k) *be a solution sequence to (1.12) satisfying* $\lambda_k \to \lambda_0$*. Then, passing to a subsequence, we have the following properties:*

(i) v_k *is uniformly bounded in* Ω.

(ii) $\sup_\Omega v_k \to -\infty$.

(iii) $\lambda_0 = \lambda_\gamma^* l$ *for some* $l \in \mathbf{N}$*, and there exist* $x_j^* \in \Omega$ *and* $x_k^{(j)}$ *for all* $1 \leq j \leq l$*, such that the following (a)-(e) hold:*

(a) $\mathcal{S} = \{x_j^*\}_{j=1}^l = \{x_0 \in \Omega \mid$ *there are* $x_k \in \Omega$ *such that* $v_k(x_k) \to +\infty\}$.

(b) $\frac{1}{2}\nabla R(x_j^*) + \sum_{i \neq j} \nabla_x G(x_i^*, x_j^*) = 0 \qquad$ *for all* $1 \leq j \leq l$.

(c) $x = x_k^{(j)}$ *is a local maximum point of* $v_k = v_k(x)$.

(d) $v_k(x_k^{(j)}) \to +\infty$ *and* $v_k \to -\infty$ *locally uniformly in* $\overline{\Omega} \setminus \mathcal{S}$ *for all* $1 \leq j \leq l$.

(e) $(v_k)_+^{\frac{n(\gamma-1)}{2}} dx \rightharpoonup \sum_{j=1}^l \lambda_\gamma^* \delta_{x_j^*}(dx) \qquad$ *in* $\mathcal{M}(\Omega)$.

Here, $G = G(x, x')$ *denotes the Green function of* $-\Delta$ *on* Ω *with the Drichlet boundary condition and*

$$R(x) = [G(x, x') - \Gamma(x - x')]_{x'=x}$$

for

$$\Gamma(x) = \frac{1}{\omega_{n-1}(n-2)|x^{n-2}|},$$

with ω_{n-1} *standing for the area of the boundary of the unit ball in* $\mathbf{R}^n$.

This paper is composed of four sections. Theorems 1 and 2 are proven in Section 2 and 3, respectively. Sketch of the proof of Theorem 3 is described in Section 4. In the following, C_i $(i = 1, 2, \cdots)$ denote positive constants whose subscripts are renewed in each section.

2 Proof of Theorem 1

In this section, we shall assume that $n \geq 3$ and $\gamma \in \left(1, \frac{n+2}{n-2}\right)$.

In order to show Theorem 1, we shall provide several lemmas.

The following lemma is shown similarly to [21].

Lemma 1 *For any* $R > 0$ *and* $A > 0$, *there exists a number* $C_1 = C_1(\gamma, R, A) > 0$ *such that*

$$\inf_{B_{R/4}} v \leq -C_1 \tag{2.1}$$

for all solutions $v \in C^2(B_R) \cap C(\overline{B_R})$ *to*

$$\begin{cases} -\Delta v = v_+^\gamma & \text{in } B_R \\ v(x_0) = 1 & \text{for some } x_0 \in B_{R/2} \\ v \leq A & \text{in } B_R. \end{cases} \tag{2.2}$$

Next, we show a uniform estimate which is crucial to obtain the boundedness from above of the solution to (1.1) with $\Omega = \mathbf{R}^n$.

Lemma 2 *There are* $C_0 = C_0(n, \gamma) > 0$ *and* $\delta_0 = \delta_0 > 0$ *such that*

$$\max_{B_{1/4}} v \leq C_0 \tag{2.3}$$

for all solutions $v \in C^2(B_1)$ *to*

$$\begin{cases} -\Delta v = v_+^\gamma & \text{in } B_1 \\ \int_{B_1} v_+^{\frac{n(\gamma-1)}{2}} < \delta_0 \end{cases} \tag{2.4}$$

Proof. If the assertion is false, then there exists a sequence $\{v_k\} \subset C^2(B_1)$ such that

$$\begin{cases} -\Delta v_k = (v_k)_+^\gamma & \text{in } B_1 \\ \int_{B_1} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \leq \frac{1}{k} \\ \max_{\overline{B_{1/4}}} v_k \geq k. \end{cases} \tag{2.5}$$

For each k, we can take $h_k \in C^2(B_1)$ and $y_k \in B_{1/2}$ such that

$$h_k(y) = \left(\frac{1}{2} - r\right)^q v_k(y), \qquad h_k(y_k) = \max_{B_{1/2}} h_k(y), \tag{2.6}$$

where $q = \frac{2}{\gamma - 1}$ and $r = |y|$. It follows from (2.5)-(2.6) that

$$\begin{aligned} h_k(y_k) &= \left(\frac{1}{2} - r_k\right)^q v_k(y_k) \geq \max_{B_{1/4}} \left(\frac{1}{2} - r\right)^q v_k(y) \\ &\geq \left(\frac{1}{4}\right)^q \max_{B_{1/4}} v_k(y) \geq \left(\frac{1}{4}\right)^q k \end{aligned} \tag{2.7}$$

for all k, where $r_k = y_k$.

Here, we consider the following function for each k:

$$w_k(y) = \mu_k^q v_k(y_k + \mu_k y) \tag{2.8}$$

with

$$\sigma_k = \frac{1}{2} - r_k, \qquad d_k^q = h_k(y_k) = \sigma_k^q v_k(y_k), \qquad \mu_k = \sigma_k/d_k. \tag{2.9}$$

We have

$$\frac{1}{2} - |y| \geq \frac{1}{2} - (|y_k| + |y - y_k|) = \left(\frac{1}{2} - r_k\right) - |y - y_k| \geq \sigma_k - \frac{\sigma_k}{2} = \frac{\sigma_k}{2}$$

for all $y \in B_{\sigma_k/2}(y_k)$, and hence

$$d_k^q = h_k(y_k) \geq \left(\frac{1}{2} - |y|\right)^q v_k(y) \geq \left(\frac{\sigma_k}{2}\right)^q v_k(y) \tag{2.10}$$

for all $y \in B_{\sigma_k/2}(y_k)$.

Noting that the function $w_k = w_k(y)$ defined by (2.8) has the scale invariance, we find

$$\begin{cases} -\Delta w_k = (w_k)_+^\gamma & \text{in } B_{d_k/2} \\ \int_{B_{d_k/2}} (w_k)_+^{\frac{n(\gamma-1)}{2}} dx = \int_{B_{\sigma_k/2}(y_k)} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \leq \frac{1}{k} & \\ w_k(0) = \mu_k^q v_k(y_k) = 1 & \\ w_k \leq 2^q & \text{in } B_{d_k/2} \end{cases} \tag{2.11}$$

by using (2.5), (2.9) and (2.10). It is also clear that $d_k \to +\infty$ by (2.7). Thus Lemma 1 and the elliptic regularity guarantee that there exist a subsequence, still denoted by $\{w_k\}$, and $\tilde{w} \in C^2(\mathbf{R}^n)$ such that

$$w_k \to \tilde{w} \qquad \text{in } C^2_{loc}(\mathbf{R}^n), \tag{2.12}$$

$$\begin{cases} -\Delta \tilde{w} = 0 & \text{in } \mathbf{R}^n \\ \tilde{w}(0) = 1 & \\ \tilde{w} \leq 2^q & \text{in } \mathbf{R}^n. \end{cases} \tag{2.13}$$

Since $\tilde{w} = \tilde{w}(x)$ is harmonic and bounded from above in $\mathbf{R}^n$ because of (2.13), it holds that

$$\tilde{w} \equiv 1 \qquad \text{in } \mathbf{R}^n$$

by Liouville's theorem, see [10], and hence (2.12) shows that $w_k \to 1$ in $C_{loc}(\mathbf{R}^n)$. This contradicts to the second of (2.11). ∎

Proposition 1 *Any classical solution to (1.1) with $\Omega = \mathbf{R}^n$ is bounded from above.*

Proof. Let $v = v(x)$ be a classical solution to (1.1) with $\Omega = \mathbf{R}^n$. Then there exists $R > 0$ such that

$$\int_{\mathbf{R}^n \setminus B_R} v_+^{\frac{n(\gamma-1)}{2}} < \delta_0$$

because of the constraint of (1.1), where δ_0 is as in Lemma 2. Therefore it follows that

$$\sup_{\mathbf{R}^n \setminus B_{R+1}} v \le C_0$$

from Lemma 2, where C_0 is a positive constant appered there. Hence the assertion holds. ∎

By virtue of Proposition 1, operating (1.1) with $(-\Delta)^{-1}$ is justified.

Lemma 3 *There exist positive numbers c_γ and c'_γ such that any nontrivial and classical solution $v = v(x)$ to (1.1) with $\Omega = \mathbf{R}^n$ has the relation*

$$v(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbf{R}^n} |x-y|^{2-n} v_+^\gamma(y)dy - c_\gamma \tag{2.14}$$

Moreover, we have the asymptotic profile

$$v(x) = -c_\gamma + c'_\gamma |x|^{2-n} + o(|x|^{2-n}), \qquad |x| \gg 1, \tag{2.15}$$

and especially the nonnegative part $v_+ = v_+(x)$ has a compact support.

Proof. We introduce the function $w = w(x)$ defined by

$$0 \le w(x) = \frac{1}{(n-2)\omega_{n-1}} \int_{\mathbf{R}^n} |x-y|^{2-n} v_+^\gamma(y)dy. \tag{2.16}$$

We shall show that (2.16) is well-defined, and that

$$\lim_{|x| \to +\infty} w(x) = 0. \tag{2.17}$$

It follows that

$$v_+ \in L^s(\mathbf{R}^n) \qquad \text{for any } s \in \left[\tfrac{n(\gamma-1)}{2}, \infty\right], \tag{2.18}$$

from the constraint of (1.1) and Proposition 1. We fix $R > 0$ and represent w as

$$0 \le w(x) = \frac{1}{(n-2)\omega_{n-1}} (w_1(x) + w_2(x)),$$

$$w_1(x) = \int_{|y-x| \ge R} |x-y|^{2-n} v_+^\gamma(y)dy, \qquad w_2(x) = \int_{|y-x| < R} |x-y|^{2-n} v_+^\gamma(y)dy.$$

Since $\gamma(n-1) \in \left[\frac{n(\gamma-1)}{2}, \infty\right)$ for $n \geq 3$, we have

$$\begin{aligned}0 \leq w_2(x) &\leq \left(\int_{|z|<R} |z|^{1-n}\right)^{\frac{n-2}{n-1}} \left(\int_{|z|<R} v_+^{\gamma(n-1)}(x-z)\right)^{\frac{1}{n-1}} \\ &\leq C_2(n,R)\|v_+\|^{\gamma}_{L^{\gamma(n-1)}(B(x,R))} \to 0 \qquad \text{as } |x| \to +\infty \end{aligned} \tag{2.19}$$

by (2.18). The term w_1 is estimated by

$$\begin{aligned}&0 \leq w_1(x) \\ &\leq \begin{cases} R^{2-n}\int_{|z|\geq R} v_+^{\gamma}(x-z)dz & \text{if } \gamma \in \left(1, \frac{n}{n-2}\right] \\ \left(\int_{|z|\geq R}|z|^{-n\left(1+\frac{2}{(n-2)\gamma-n}\right)}dz\right)^{\frac{(n-2)\gamma-n}{n(\gamma-1)}} & \\ \quad\times\left(v_+^{\frac{n(\gamma-1)}{2}}dz\right)^{\frac{2\gamma}{n(\gamma-1)}} & \text{if } \gamma \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right) \end{cases} \\ &\leq \begin{cases} R^{2-n}\|v_+\|^{\gamma}_{\gamma} & \text{if } \gamma \in \left(1, \frac{n}{n-2}\right] \\ R^{-\frac{1}{\gamma-1}}C_3(n,\gamma)\|v_+\|^{\gamma}_{\frac{n(\gamma-1)}{2}} & \text{if } \gamma \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right) \end{cases}\end{aligned} \tag{2.20}$$

Combining (2.18)-(2.20), and noting that $\gamma \in \left[\frac{n(\gamma-1)}{2}, \infty\right)$ for $\gamma \in \left(1, \frac{n}{n-2}\right]$, we see that (2.16) is well-defined, and that

$$0 \leq \limsup_{|x|\to+\infty} w(x) \leq \begin{cases} C_4(n,\gamma)R^{2-n} & \text{if } \gamma \in \left(1, \frac{n}{n-2}\right] \\ C_5(n,\gamma)R^{\frac{1}{\gamma-1}} & \text{if } \gamma \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right), \end{cases}$$

which implies (2.17) since $R > 0$ is arbitrary.

We have now

$$-\Delta(v-w) = 0 \quad \text{in } \mathbf{R}^n, \qquad \sup_{\mathbf{R}^n}(v-w) < +\infty$$

by (2.16) and Proposition 1. Then, Liouville's thorem, see [10], guarantees that there exists $c_\gamma \in \mathbf{R}^n$ such that $v - w = c_1$. We claim that $c_1 < 0$. If this is not the case then

$$-\Delta v = v^{\gamma}, v \geq 0 \qquad \text{in } \mathbf{R}^n,$$

which is impossible because of $1 < \gamma < \frac{n+2}{n-2}$ and the result from [8]. Thus we obtain (2.14) for $c_\gamma = -c_1 > 0$.

It holds by (2.14) and the dominated convergence theorem that

$$\begin{aligned}|x|^{n-2}(v(x) - c_\gamma) &= w(x) \\ &= \frac{1}{(n-2)\omega_{n-1}}\int_{\mathbf{R}^n}\frac{|x|^{n-2}}{|x-y|^{n-2}}v_+^{\gamma}(y)dy \\ &\to \frac{1}{(n-2)\omega_{n-1}}\int_{\mathbf{R}^n} v_+^{\gamma}dx\end{aligned}$$

as $|x| \to +\infty$, which implies (2.15) for $c'_\gamma = \frac{1}{(n-2)\omega_{n-1}}\int_{\mathbf{R}^n} v_+^{\gamma}dx$. ∎

Proof of Theorem 1: First, we shall show the radial symmetricity of the solution $v = v(x)$ to (1.1) with $\Omega = \mathbf{R}^n$. To show this, we have only to show that $w = w(x)$ defined by (2.16) also satisfies the same property. We introduce the function

$$f(t) = (t - c_\gamma)_+, \tag{2.21}$$

where $c_\gamma > 0$ is a positive constant in (2.14). Then, it holds that

$$\begin{cases} -\Delta w = f(w) & \text{in } \mathbf{R}^n \\ w > 0 \\ \lim_{|x| \to +\infty} w(x) = 0 \end{cases} \tag{2.22}$$

by virtue of Lemma 3. Noting (2.21) and the asymptotic profile (2.15), we can apply the result from [9] and conclude that the solution $w = w(x)$ to (2.22) has the desired property. Namely, there exist a point $x_0 \in \mathbf{R}^n$ and a function $V = V(r)$ defined on $[0, +\infty)$ such that

$$v(x) = V(r), \quad v(x_0) = V(0) = \sup_{x \in \mathbf{R}^n} v(x), \quad V'(r) < 0 \quad (\text{for } r > 0), \tag{2.23}$$

where $r = |x - x_0|$.

We can readily deduce the remainder of the assetions of Theorem 1 from (2.23) and some direct computations. The proof is complete. ∎

3 Proof of Theorem 2

In this section, we shall assume that $n \geq 3$ and $\gamma \in \left(1, \frac{n+2}{n-2}\right)$, again.

We begin with an *a priori* bound of the solution to (2.4).

Lemma 4 *For any $\delta \in (0, \lambda_\gamma^*)$, we have a constant $C_\delta = C_\delta(n, \gamma, \delta) > 0$ such that*

$$\max_{B_{1/4}} v \leq C_\delta \tag{3.1}$$

for any solution $v = v(x)$ to (2.4) with $\delta_0 = \delta$.

Proof. Fix $\delta \in (0, \lambda_\gamma^*)$ and suppose that the assertion is false. Then we can discuss as in the proof of Lemma 2 and find that there exists $w \in C^2(\mathbf{R}^n)$ such that

$$\begin{cases} -\Delta w = w_+^\gamma & \text{in } \mathbf{R}^n \\ \int_{\mathbf{R}^n} w_+^{\frac{n(\gamma-1)}{2}} dx \leq \delta < \lambda_\gamma^* \\ w(0) = 1 \\ w \leq 2^q, \quad q = \frac{2}{\gamma - 1} & \text{in } \mathbf{R}^n, \end{cases}$$

which is a contradiction by Theorem 1. ∎

One can see that Theorem 2 is a direct consequence of the following lemma.

Lemma 5 *Let T be a positive constant. Then we have $C_1 = C_1(n, \gamma) > 0$ and $C_2 = C_2(n, \gamma, T) > 0$ such that*

$$v(0) + C_1 \inf_{B_1} v \leq C_2 \tag{3.2}$$

for any solution $v = v(x) \in C^2(B_1)$ *to*

$$\begin{cases} -\Delta v = v_+^{\gamma} & \text{in } B_1 \\ \int_{B_1} v_+^{\frac{n(\gamma-1)}{2}} dx \leq T. \end{cases} \tag{3.3}$$

Proof. Suppose that the assertion does not hold. Then for any $\hat{C} > 0$, there exists a sequence $\{v_k\} \subset C^2(B_1)$ such that

$$\begin{cases} -\Delta v_k = (v_k)_+^{\gamma} & \text{in } B_1 \\ \int_{B_1} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \leq T \\ v_k(0) + \hat{C} \inf_{B_1} v_k \geq k. \end{cases} \tag{3.4}$$

It is obvious that

$$v_k(0) \geq \frac{k}{1+\hat{C}} \to +\infty \tag{3.5}$$

as $k \to \infty$.

Here, we use $h_k \in C^2(B_1)$, $y_k \in B_{1/2}$, $w_k = w_k(y)$, σ_k, d_k and μ_k that are taken in the proof of Lemma 2, see (2.6) and (2.8)-(2.9). Then it holds that

$$d_k \geq (v_k(0))^{1/q} \to +\infty. \tag{3.6}$$

by (3.5). We have also (2.10) for all $y \in B_{\sigma_k/2}(y_k)$, and so

$$w_k \leq 2^q \qquad \text{in } B_{d_k/2}(y_k). \tag{3.7}$$

Similarly to the proof of Lemma 2, we deduce

$$\begin{cases} -\Delta w_k = (w_k)_+^{\gamma} & \text{in } B_{d_k/2} \\ \int_{B_{d_k/2}} (w_k)_+^{\frac{n(\gamma-1)}{2}} dx = \int_{B_{\sigma_k/2}(y_k)} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \leq T & \\ w_k(0) = 1 & \\ w_k \leq 2^q & \text{in } B_{d_k/2} \end{cases}$$

from (3.4) and (3.7). Therefore, we can extract a subsequence, still denoted by $\{w_k\}$, and a function $\tilde{w} \in C^2(\mathbf{R}^n)$ such that

$$w_k \to \tilde{w} \qquad \text{in } C^2_{loc}(\mathbf{R}^n), \tag{3.8}$$

$$\begin{cases} -\Delta \tilde{w} = 0 & \text{in } \mathbf{R}^n \\ \int_{\mathbf{R}^n} \tilde{w}_+^{\frac{n(\gamma-1)}{2}} dx \leq T & \\ \tilde{w}(0) = 1 & \\ \tilde{w} \leq 2^q & \text{in } \mathbf{R}^n, \end{cases} \tag{3.9}$$

where we have used (3.6), Lemma 1 and the elliptic regularity.

We may assume $T \geq \lambda_\gamma^*$ thanks to Theorem 1. Noting the third and fourth properties of (3.9), we have (1.4) for some $x_0 \in \mathbf{R}^n$ and $\mu = \mu_0 \in [1, 2]$. In particular, it holds that

$$w(0) = 1, \qquad \lim_{|x| \to +\infty} w(x) \leq -C_3$$

for some $C_3 = C_3(n,\gamma) > 0$. Consequently, there exist $C_4 = C_4(n,\gamma) > 0$ and $R = R(n,\gamma) \gg 1$ such that

$$w(0) + C_4 \inf_{\partial B_R} w < 0. \tag{3.10}$$

Hence it follows from (3.8) and (3.10) that

$$w_k(0) + C_4 \inf_{\partial B_R} w_k < 0. \tag{3.11}$$

for $k \gg 1$.

Noting that v_k is super-harmonic, and that $B(y_k, \mu_k R) \subset B_1$ for $k \gg 1$ by (3.6). Then we obtain

$$\begin{aligned} v_k(0) + C_4 \inf_{B_1} v_k &\le v_k(y_k) + C_4 \inf_{\partial B(y_k,\mu_k R)} v_k \\ &= \mu_k^{-q}\left(w_k(0) + C_4 \inf_{\partial B_R} w_k \right) < 0 \end{aligned}$$

for $k \gg 1$ by virtue of the scale invariance and (3.11). However, this is contrary to (3.4) if $\hat{C} \ge C_4$, since $v_k(0) > 0$ by (3.4). ∎

Proof of Theorem 2: Let Ω be a bounded domain, fix any positive number T and compact set $K \subset \Omega$, and suppose that $v = v(x)$ is a classical solution to (1.1) and satisfies (1.9). Then we have $\mu_0 = \mu_0(K) > 0$ and $x_0 \in K$ such that

$$\bigcup_{x \in K} B(x, \mu_0) \subset \Omega, \qquad v(x_0) = \sup_K v.$$

We introcude the function

$$w(x) = \mu_0^q v(x_0 + \mu_0 x)$$

for $x \in B_1$ and $q = \frac{2}{\gamma - 1}$. By the scale invariance, it holds that

$$v(x_0) + C \inf_{\Omega} v \le v(x_0) + C \inf_{B(x_0,\mu_0)} v = \mu_0^{-q}(w(0) + C \inf_{B_1} w), \tag{3.12}$$

for any $C > 0$, and that $w = w(x)$ satisfies (3.3). Hence Lemma 5 yields $C_5 = C_5(n,\gamma) > 0$ and $C_6 = C_6(n,\gamma,T)$ such that

$$w(0) + C_5 \inf_{B_1} w \le C_6. \tag{3.13}$$

Inequality (1.8) follows from (3.12) and (3.13) as $C_1 = C_5$ and $C_2 = \mu_0^{-q} C_6$. ∎

4 Proof of Theorem 3 (Sketch)

In this section, we shall assume that $\gamma \in \left(\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ and $n \ge 3$. Also, we shall denote a subsequence of the sequence by the same notation without notice.

Proof of Theorem 3 is reduced to those of the following two propositions:

Proposition 2 *Assume that* $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ *and* $n \geq 3$. *Let* $\Omega \subset \mathbf{R}^n$ *be a bounded domain with smooth boundary* $\partial\Omega$ *and* $\{v_k\}$ *be a sequence of the classical solutions satisfying (1.10) for some* $T > 0$. *Then there exists a subsequence, still denoted by the same symbol* $\{v_k\}$, *such that the following alternatives occur:*

(i) $\{v_k\}$ *is locally uniformly bounded.*

(ii) $v_k \to -\infty$ *locally uniformly in* Ω.

(iii) *There exists a finite set* $\mathcal{S} = \{x_i\}_{i=1}^m$ *such that* $v_k \to -\infty$ *locally uniformly in* $\Omega \setminus \mathcal{S}$ *and that*

$$(v_k)_+^{\frac{n(\gamma-1)}{2}} dx \rightharpoonup \sum_{i=1}^{m} \alpha_*(x_i)\delta_{x_i}(dx)$$

in $\mathcal{M}(\Omega)$ *with* $\alpha_*(x_i) \geq \lambda_\gamma^*$ *for all* $i = 1, \cdots, m$.

Proposition 3 *In the alternative (iii) of Proposition 2, it holds that* $\alpha_*(x_i) = l_i\lambda_\gamma^*$ *for some* $l_i \in \mathbf{N}$ *and for all* $i = 1, \cdots, m$.

Proof of Proposition 2: Since $\{(v_k)_+^{\frac{n(\gamma-1)}{2}}\}$ is bounded in $L^1(\Omega)$, there exist a subsequence $\{v_k\}$ and a bounded non-negative measure μ such that

$$(v_k)_+^{\frac{n(\gamma-1)}{2}} dx \rightharpoonup \mu \qquad \text{in } \mathcal{M}(\Omega), \tag{4.1}$$

where $\mathcal{M}(\Omega)$ stands for the space of measure. Set

$$\Sigma = \{x \in \Omega \mid \mu(\{x\}) \geq \lambda_\gamma^*\}$$
$$\mathcal{S} = \{x \in \Omega \mid \text{there exists } \{x_k\} \subset \Omega \text{ such that } x_k \to x \text{ and } v_k(x_k) \to +\infty.\}.$$

First, we claim

$$\Sigma = \mathcal{S}. \tag{4.2}$$

Suppose that $x_0 \notin \Sigma$. Then there exists $0 < r_0 \ll 1$ such that

$$\mu(B(x_0, r_0)) < \lambda_\gamma^* \tag{4.3}$$

because of the property of the bounded non-negative measure. Hence we obtain $\delta_0 \in (0, \lambda_\gamma^*)$ such that

$$\int_{B(x_0,r_0)} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \leq \delta_0$$

for $k \gg 1$ by (4.1) and (4.3). Putting

$$w_k(x) = r_0^q v_k(x_0 + r_0 x)$$

for $x \in B_1$ and $q = \frac{2}{\gamma-1}$, we see that w_k satisfies

$$\begin{cases} -\Delta w_k = (w_k)_+^\gamma & \text{in } B_1 \\ \int_{B_1} (w_k)_+^{\frac{n(\gamma-1)}{2}} dx \leq \delta_0 \end{cases}$$

for $k \gg 1$. Consequently, Lemma 4 assures that there exists $C_{\delta_0} = C_{\delta_0}(n, \gamma, \delta_0) > 0$ such that

$$\max_{B_{1/4}} w_k \leq C_{\delta_0}$$

for $k \gg 1$, which implies

$$\max_{B(x_0, r_0/4)} v_k \leq r_0^{-q} C_{\delta_0}$$

for $k \gg 1$. Thus we have $\mathcal{S} \subset \Sigma$. In turn, suppose that $x_0 \notin \mathcal{S}$. From the definition of $\mathcal{S}$, it is clear that there exists $0 < r_0 \ll 1$ such that

$$\sup_k \|(v_k)_+\|_{L^\infty(B(x_0, r_0))} < +\infty$$

for some subsequence $\{v_k\}$. Hence we obtain

$$\lim_{r \downarrow 0} \limsup_{k \to \infty} \int_{B(x_0, r_0)} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = 0. \tag{4.4}$$

We deduce from (4.1) and (4.4) that $\mu(\{x_0\}) = 0$, and therefore $x_0 \notin \Sigma$. Thus we have $\Sigma \subset \mathcal{S}$, and hence (4.2).

Next, we shall show that $\mathcal{S} = \emptyset$ implies (i) or (ii). Assume that $\mathcal{S} = \emptyset$ and fix an open set ω satisfying $\overline{\omega} \subset \Omega$. Similarly to the proof of (4.2), we deduce that there exists $C_1 = C_1(n, \gamma, \omega) > 0$ such that

$$\sup_k \|(v_k)_+\|_{L^\infty(\omega)} \leq C_1. \tag{4.5}$$

Let $v_{1,k}$ be a solution to

$$\begin{cases} -\Delta v_{1,k} = (v_k)_+^\gamma & \text{in } \omega \\ v_{1,k} = 0 & \text{on } \partial\omega. \end{cases}$$

It holds that $v_{1,k} \geq 0$ in ω by the maximum principle, and that $\{v_{1,k}\}$ is uniformly bounded in ω because of (4.5) and the elliptic regularity. In other words, there exists $C_2 = C_2(n, \gamma, \omega) > 0$ such that

$$0 \leq v_{1,k} \leq C_2 \qquad \text{in } \omega. \tag{4.6}$$

Hence $\tilde{v}_k = v_k - v_{1,k}$ is harmonic and bounded from above in ω. Since ω is arbitrary, we use the Harnack principle to the harmonic function and find that $\{\tilde{v}_k\}$ is locally uniform bounded in Ω, or otherwise $\tilde{v}_k \to -\infty$ locally uniformly in Ω. Noting inequality (4.6), we have (i) or (ii) in each cases.

Finally, we shall show that $\mathcal{S} \neq \emptyset$ implies (iii). Since $\mathcal{S} = \{x_i\}_{i=1}^m$ is finite, we perfome the argument similar to above and find that $\{v_k\}$ is bounded in $L^\infty_{loc}(\Omega \setminus \mathcal{S})$, or otherwise $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$. We now claim that the former does not hold. To show this claem, we suppose the contrary and take $r_1 > 0$ such that $B(x_1, r_1) \cap \mathcal{S} = \{x_1\}$ which is possible by the finiteness of $\mathcal{S}$. Then there exists $C_3 = C_3(n, \gamma, x_1, r_1) > 0$ such that

$$v_k \geq -C_3 \qquad \text{on } \partial B(x_1, r_1). \tag{4.7}$$

Let z_k be a solution to

$$\begin{cases} -\Delta z_k = (v_k)_+^\gamma & \text{in } B(x_1, r_1) \\ z_k = -C_3 & \text{on } \partial B(x_1, r_1). \end{cases}$$

We obtain $z_k \le v_k$ in $B(x_1, r_1)$, and

$$z_k(x)dx \rightharpoonup \alpha\delta_{x_1}(dx) + f(x)dx$$

in $\mathcal{M}(\overline{B(x_1, r_1)})$ with

$$\alpha \ge \lambda_\gamma^* \quad \text{and} \quad 0 \le f \in L^1(B(x_1, r_1)),$$

and therefore $z_k \to z$ locally uniformly in $\overline{B(x_1, r_1)} \setminus \{x_1\}$ with

$$z(x) \ge \frac{\lambda_\gamma^*}{\omega_{n-1}(n-2)|x - x_1|^{n-2}} - O(1)$$

for $x \in \overline{B(x_1, r_1)} \setminus \{x_1\}$. Then Fatou's lemma assures

$$\begin{aligned} +\infty = \int_{B(x_1, r_1)} z_+^{\frac{n(\gamma-1)}{2}} dx &\le \liminf_k \int_{B(x_1, r_1)} (z_k)_+^{\frac{n(\gamma-1)}{2}} dx \\ &\le \liminf_k \int_{B(x_1, r_1)} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx < +\infty \end{aligned}$$

because of the assumption $\gamma \in \left[\frac{n}{n-2}, \frac{n+2}{n-2}\right)$ and the constraint of (1.10). This inequality is a contradiction. Thus we obtain $v_k \to -\infty$ locally uniformly in $\Omega \setminus \mathcal{S}$. The proof is complete. ∎

Proof of Proposition 3 is done similarly to [13]. More precisely, it is reduced to the following lemmas.

Lemma 6 *Given $R > 0$, we assume that $v_k = v_k(x)$ satisfies*

$$-\Delta v_k = (v_k)_+^\gamma \qquad \text{in } B_R, \tag{4.8}$$

$$\max_{\overline{B_R}} v_k \to +\infty \quad \text{and} \quad \max_{\overline{B_R} \setminus B_r} v_k \to -\infty \qquad \text{for any } r \in (0, R), \tag{4.9}$$

$$\lim_{k\to\infty} \int_{B_R} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = \alpha \qquad \text{for some } \alpha > 0, \tag{4.10}$$

$$\sup_k \sup_{x \in B_R} v_k(x)|x|^q \le C_4 \qquad \text{for some } C_4 > 0, \tag{4.11}$$

where $q = \frac{2}{\gamma-1}$. Then, $\alpha = \lambda_\gamma^*$ and there exist $C_5 = C_5(\cdots) > 0$ and $k_0 \in \mathbf{N}$ such that

$$v_k \le 0 \qquad \text{in } \overline{\Omega} \setminus B_{C_5\delta_k}$$

for all $k \ge k_0$ with $\delta_k^q = \max_{\overline{B_R}} v_k$.

Lemma 7 *Given $R > 0$, we assume that $v_k = v_k(x)$ satisfies (4.8)-(4.10) and there is $T > 0$, independent of k, such that*

$$\int_{B_R} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx \le T \tag{4.12}$$

for all k. *Then, passing to a subsequence, we have* $\{x_k^{(j)}\}_{j=0}^{m-1} \subset B_R$, $\{l_k^{(j)}\}_{j=0}^{m-1} \subset$ $\mathbf{N}$ *and* $m \in \mathbf{N}$ *with* $x_k^{(j)} \to 0$, $l_k^{(j)} \to \infty$ *and* $1 \leq m \leq T/\lambda_\gamma^*$ *such that the following (4.13)-(4.17) hold:*

$$v_k(x_k^{(j)}) = \max_{|x-x_k^{(j)}| \leq l_k^{(j)}\delta_k^{(j)}} v_k(x) \to +\infty \tag{4.13}$$

for all $0 \leq j \leq m-1$,

$$B(x_k^{(i)}, 2l_k^{(i)}\delta_k^{(i)}) \cap B(x_k^{(j)}, 2l_k^{(j)}\delta_k^{(j)}) = \emptyset \tag{4.14}$$

for all k *and* $0 \leq i, j \leq m-1$ *satisfying* $i \neq j$,

$$\left.\frac{\partial}{\partial t} v_k(ty + x_k^{(j)})\right|_{t=1} < 0 \tag{4.15}$$

for all k, $0 \leq j \leq m-1$ *and* y *satisfying* $2r_\gamma^*\delta_k^{(j)} \leq |y| \leq 2l_k^{(j)}\delta_k^{(j)}$,

$$\lim_{k\to} \int_{B(x_k^{(j)}, 2l_k^{(j)}\delta_k^{(j)})} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = \int_{B(x_k^{(j)}, l_k^{(j)}\delta_k^{(j)})} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = \lambda_\gamma^* \tag{4.16}$$

for all $0 \leq j \leq m-1$, *and*

$$\max_{\overline{B_R}} \left\{ v_k(x) \min_{0 \leq j \leq m-1} |x - x_k^{(j)}|^q \right\} \leq C_6 \tag{4.17}$$

for all k *and for some* $C_6 > 0$ *independent of* k, *where* $(\delta_k^{(j)})^q = v_k(x_k^{(j)})$, $q = \frac{2}{\gamma-1}$, *and* r_γ^* *is as in Theorem 1.*

Lemma 8 *Given* $R > 0$, *we assume that* $v_k = v_k(x)$ *satisfies (4.8)-(4.10), (4.12), and that there exist* $\{x_k^{(j)}\}_{j=0}^{m-1}$ *and* $\{r_k^{(j)}\}_{j=0}^{m-1}$, $m \geq 1$, $r_k^{(j)} > 0$, *such that the following (4.18)-(4.22) hold:*

$$v_k(x_k^{(j)}) = \to +\infty \tag{4.18}$$

for all $0 \leq j \leq m-1$,

$$\lim_{k\to\infty} \frac{r_k^{(j)}}{\delta_k^{(j)}} = +\infty \tag{4.19}$$

for all $0 \leq j \leq m-1$,

$$B(x_k^{(i)}, r_k^{(i)}) \cap B(x_k^{(j)}, r_k^{(j)}) = \emptyset \tag{4.20}$$

for all k *and* $0 \leq i, j \leq m-1$ *satisfying* $i \neq j$,

$$\max_{\overline{B_R} \setminus \cup_{j=0}^{m-1} B(x_k^{(j)}, r_k^{(j)})} \left\{ v_k(x) \min_{0 \leq j \leq m-1} |x - x_k^{(j)}|^q \right\} \leq C_7 \tag{4.21}$$

for all k *and for some* $C_7 > 0$ *independent of* k, *and*

$$\lim_{k\to\infty} \int_{B(x_k^{(j)}, 2r_k^{(j)})} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = \lim_{k\to\infty} \int_{B(x_k^{(j)}, r_k^{(j)})} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = \beta_j \tag{4.22}$$

for some $\beta_j > 0$, $0 \leq j \leq m-1$. *Then it holds that*

$$\lim_{k\to\infty} \int_{B_R} (v_k)_+^{\frac{n(\gamma-1)}{2}} dx = \sum_{j=0}^{m-1} \beta_j. \tag{4.23}$$

Proposition 3 is obtained by combining Lemmas 6-8. We will be able to find their rigorous proofs in the forthcoming paper.

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